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## Sufficiency of cut-generating functions

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**CORE**

DISCUSSION PAPER

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**Sufficiency of cut-generating functions**

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**Abstract**

This note settles an open problem about cut-generating functions, a concept that has its origin in the work of Gomory and Johnson from the 1970's and has received renewed attention in recent years.

**Keywords:** mixed integer programming, separation, corner polyhedron, intersection cuts.

**Mathematics Subject Classification:** 90C11, 90C26

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# 1 Introduction

We consider sets of the form

$$X = X(R, S) := \{x \in \mathbb{R}_+^n : Rx \in S\}, \quad (1a)$$

$$\text{where } \begin{cases} R = [r_1 \dots r_n] \text{ is a real } q \times n \text{ matrix,} \\ S \subset \mathbb{R}^q \text{ is a nonempty closed set with } 0 \notin S. \end{cases} \quad (1b)$$

This model has been studied in [Joh81] and [CCD<sup>+</sup>13]. It appears in cutting plane theory [Gom69, GJ72, ALWW07, JSRF06] where the goal is to generate inequalities that are valid for  $X$  but not for the origin. Such cutting planes are well-defined [CCD<sup>+</sup>13, Lemma 2.1] and can be written as

$$c^\top x \geq 1. \quad (2)$$

Let  $S \subset \mathbb{R}^q$  be a given nonempty closed set with  $0 \notin S$ .  $S$  is assumed to be fixed in this paragraph. [CCD<sup>+</sup>13] introduce the notion of a *cut-generating function*: This is any function  $\rho : \mathbb{R}^q \mapsto \mathbb{R}$  that produces coefficients  $c_j := \rho(r_j)$  of a cut (2) valid for  $X(R, S)$  for any choice of  $n$  and  $R = [r_1 \dots r_n]$ . It is shown in [CCD<sup>+</sup>13] that cut-generating functions enjoy significant structure. For instance, the minimal ones are sublinear and are closely related to  $S$ -free neighborhoods of the origin. We say that a closed convex set is  *$S$ -free* if it contains no point of  $S$  in its interior. For any minimal cut-generating function  $\rho$ , there exists a closed convex  $S$ -free set  $V \subset \mathbb{R}^q$  such that  $0 \in \text{int } V$  and  $V = \{r \in \mathbb{R}^q : \rho(r) \leq 1\}$ . A cut (2) with coefficients  $c_j := \rho(r_j)$  is called an  *$S$ -intersection cut*.

Now assume that both  $S$  and  $R$  are fixed. Noting  $X(R, S) \subset \mathbb{R}_+^n$ , we say that a cutting plane  $c^\top x \geq 1$  *dominates*  $b^\top x \geq 1$  if  $c_j \leq b_j$  for  $j \in [n]$ . A natural question is whether every cut (2) valid for  $X(R, S)$  is dominated by an  $S$ -intersection cut. [CCD<sup>+</sup>13] give an example showing that this is not always the case. However, this example has the peculiarity that  $S$  contains points that cannot be obtained as  $Rx$  for any  $x \in \mathbb{R}_+^n$ . [CCD<sup>+</sup>13] propose the following open problem: Assuming  $S \subset \text{cone } R$ , is it true that every cut (2) valid for  $X(R, S)$  is dominated by an  $S$ -intersection cut? Our main theorem shows that this is indeed the case. This generalizes the main result of [CCZ10] and Theorem 6.3 in [CCD<sup>+</sup>13].

**Theorem 1.1.** *Suppose  $S \subset \text{cone } R$ . Then any valid inequality  $c^\top x \geq 1$  separating the origin from  $X$  is dominated by an  $S$ -intersection cut.*

## 2 Proof of the Main Theorem

Our proof of Theorem 1.1 will use several lemmas. We first introduce some terminology. Given a convex cone  $K \subseteq \mathbb{R}^d$ , let  $K^\circ := \{w \in \mathbb{R}^d : u^\top w \leq 0, \forall u \in K\}$  (resp.  $K^* := \{w \in \mathbb{R}^d : u^\top w \geq 0, \forall u \in K\}$ ) denote the *polar* (resp. *dual*) of  $K$ . Let  $\sigma_W(u) := \sup_{w \in W} u^\top w$  be the *support function* of a set  $W \subseteq \mathbb{R}^d$ . A function  $\rho : \mathbb{R}^d \mapsto \mathbb{R} \cup \{+\infty\}$  is said to be *positively homogeneous* if  $\rho(\lambda u) = \lambda \rho(u)$  for all  $\lambda > 0$  and  $u \in \mathbb{R}^d$  and *subadditive* if  $\rho(u_1) + \rho(u_2) \geq \rho(u_1 + u_2)$  for all  $u_1, u_2 \in \mathbb{R}^d$ . Moreover,  $\rho$  is *sublinear* if it is both positively homogeneous and subadditive. Sublinear functions are known to be convex and it is not difficult to show that support functions are sublinear (see, e.g., [HUL04, Chapter C]). Given a closed convex neighborhood  $V$  of the origin, a *representation of  $V$*  is any sublinear function  $\rho : \mathbb{R}^q \mapsto \mathbb{R}$  such that  $V = \{r \in \mathbb{R}^q : \rho(r) \leq 1\}$ .  $S$ -intersection cuts are generated via representations of closed convex  $S$ -free neighborhoods of the origin.

Throughout this section, we assume that  $X \neq \emptyset$  and  $c^\top x \geq 1$  is a valid inequality separating the origin from  $X$ .

**Lemma 2.1.** *If  $u \in \mathbb{R}_+^n$  and  $Ru = 0$ , then  $c^\top u \geq 0$ , or, equivalently,  $c \in \mathbb{R}_+^n + \text{Im } R^\top$ .*

*Proof.* Let  $\bar{x} \in X$ . Note that  $R(\bar{x} + tu) = R\bar{x} \in S$  and  $\bar{x} + tu \geq 0$  for all  $t \geq 0$ . By the validity of  $c$ , we have  $c^\top(\bar{x} + tu) \geq 1$  for all  $t \geq 0$ . Observing  $tc^\top u \geq 1 - c^\top \bar{x}$  and letting  $t \rightarrow +\infty$  implies  $c^\top u \geq 0$  as desired. Because  $u$  is an arbitrary vector in  $\mathbb{R}_+^n \cap \text{Ker } R$ , we can write  $c \in (\mathbb{R}_+^n \cap \text{Ker } R)^*$ . The equality  $(\mathbb{R}_+^n \cap \text{Ker } R)^* = \mathbb{R}_+^n + \text{Im } R^\top$  follows from the facts  $(\mathbb{R}_+^n)^* = \mathbb{R}_+^n$ ,  $(\text{Ker } R)^* = \text{Im } R^\top$  and  $\mathbb{R}_+^n + \text{Im } R^\top$  is closed (see, e.g., [Roc70, Cor. 16.4.2]).  $\blacksquare$

Let

$$\begin{aligned} h(r) := \min \quad & c^\top x \\ & Rx = r, \\ & x \geq 0. \end{aligned} \tag{3}$$

**Remark 2.2.**  $h(r_j) \leq c_j$  for all  $j \in [n]$ .

**Lemma 2.3.**  $h$  is a piecewise-linear sublinear function on the domain cone  $R$ .

*Proof.* The domain of  $h$  must be a subset of cone  $R$  because (3) is infeasible for  $r \notin \text{cone } R$ . The dual of (3) is

$$\max \quad \begin{aligned} & r^\top y \\ & R^\top y \leq c. \end{aligned} \tag{4}$$

Let  $P := \{y \in \mathbb{R}^q : R^\top y \leq c\}$ . By Lemma 2.1,  $c = c' + c''$  where  $c' \in \mathbb{R}_+^n$  and  $c'' \in \text{Im } R^\top$ . Because  $c'' \in \text{Im } R^\top$ , there exists  $y'' \in \mathbb{R}^q$  such that  $R^\top y'' = c'' \leq c$ . Hence  $y'' \in P$  which shows that the dual LP is always feasible, strong duality holds and  $h(r) = \sigma_P(r)$  for all  $r \in \text{cone } R$ . In particular,  $h(0) = 0$  and  $h(r)$  is finite for all  $r \in \text{cone } R$ . Now let  $W$  be a finite set of points for which  $P = \text{conv } W + \text{rec } P$ . Observe that  $\text{rec } P = (\text{cone } R)^\circ$  and  $r^\top u \leq 0$  for all  $r \in \text{cone } R$  and  $u \in \text{rec } P$ . Therefore,  $h(r) = \sigma_P(r) = \sigma_W(r)$  for all  $r \in \text{cone } R$  which implies that  $h$  is piecewise-linear and sublinear on the domain  $\text{cone } R$ .  $\blacksquare$

**Lemma 2.4.** *Theorem 1.1 holds when  $\text{cone } R = \mathbb{R}^q$ .*

*Proof.* In this case,  $h$  is finite everywhere. Let  $V := \{r \in \mathbb{R}^q : h(r) \leq 1\}$ . Because the Slater condition is satisfied, we have  $\text{int } V = \{r \in \mathbb{R}^q : h(r) < 1\}$  (see, e.g., [HUL04, Prop. D.1.3.3]). Thus  $V$  is a closed convex neighborhood of the origin and  $h$  represents  $V$  by definition.

*Claim 2.1:*  $V$  is  $S$ -free. Suppose this is not the case. Let  $\bar{r} \in S$  be a point in  $\text{int } V$ . Then there exists  $\bar{x} \geq 0$  such that  $R\bar{x} = \bar{r} \in S$  and  $c^\top \bar{x} = h(\bar{r}) < 1$ . Because  $\bar{x} \in X$ , this contradicts the validity of  $c^\top x \geq 1$ .  $\square$

Therefore,  $\sum_{j=1}^n h(r_j)x_j \geq 1$  is an  $S$ -intersection cut that can be obtained from the closed convex  $S$ -free neighborhood  $V$  of the origin. By Remark 2.2,  $h(r_j) \leq c_j$  for all  $j \in [n]$ . This shows that  $\sum_{j=1}^n h(r_j)x_j \geq 1$  dominates  $c^\top x \geq 1$ .  $\blacksquare$

We now consider the case where  $\text{cone } R \subsetneq \mathbb{R}^q$ . We want to extend the definition of  $h$  to the whole of  $\mathbb{R}^q$  and show that this extension is a cut-generating function. We will first construct a function  $h'$  such that 1)  $h'$  is finite everywhere on  $\text{span } R$ , 2)  $h'$  coincides with  $h$  on  $\text{cone } R$ . If  $\dim(R) < q$ , we will further extend  $h'$  to the whole of  $\mathbb{R}^q$  by letting  $h'(r) = h'(r')$  for all  $r \in \mathbb{R}^q$ ,  $r' \in \text{span } R$ ,  $r'' \in (\text{span } R)^\perp$  such that  $r = r' + r''$ . Our proof of Theorem 1.1 will show that this procedure yields a function  $h'$  that is the desired extension of  $h$ .

Let  $r_0 \in -\text{ri}(\text{cone } R)$  where  $\text{ri}(\cdot)$  denotes the relative interior. Note that this guarantees  $\text{cone}(R \cup \{r_0\}) = \text{span } R$  since there exist  $\epsilon > 0$  and  $d := \dim(R)$  linearly independent vectors  $a_1, \dots, a_d \in \text{span } R$  such that  $-r_0 \pm \epsilon a_i \in \text{cone } R$  for all  $i \in [d]$  which implies  $\pm a_i \in \text{cone}(R \cup \{r_0\})$ . Now we define  $c_0$  as

$$c_0 := \sup_{r \in \text{cone } R} \sup_{\alpha > 0} \frac{h(r) - h(r + \alpha(-r_0))}{\alpha}. \quad (5)$$

**Lemma 2.5.**  $c_0$  is finite.

*Proof.* Any pair  $\bar{r} \in \text{cone } R$  and  $\bar{\alpha} > 0$  yields a lower bound on  $c_0$ : Our choice of  $r_0$  ensures  $\bar{r} + \bar{\alpha}(-r_0) \in \text{cone } R$  and  $c_0 \geq \frac{h(\bar{r}) - h(\bar{r} + \bar{\alpha}(-r_0))}{\bar{\alpha}}$ . To get an upper bound on  $c_0$ , consider the LPs (3) and (4). Let  $\tilde{r} \in \text{cone } R$  and  $\tilde{\alpha} \geq 0$ . Observe that  $\tilde{r} + \tilde{\alpha}(-r_0) \in \text{cone } R$  and, as in the proof of Lemma 2.3, one can show that both LPs are feasible when we plug in  $\tilde{r} + \tilde{\alpha}(-r_0)$  for  $r$ . Therefore, strong duality holds and  $h(\tilde{r} + \tilde{\alpha}(-r_0)) = \sigma_P(\tilde{r} + \tilde{\alpha}(-r_0))$  where  $P := \{y \in \mathbb{R}^q : R^\top y \leq c\}$  is the feasible region of (4). Let  $W$  be a finite set of points for which  $P = \text{conv } W + \text{rec } P$ . Because  $\text{rec } P = (\text{cone } R)^\circ$ , we have  $(\tilde{r} + \tilde{\alpha}(-r_0))^\top u \leq 0$  for all  $u \in \text{rec } P$ . This implies  $\sigma_P(\tilde{r} + \tilde{\alpha}(-r_0)) = \sigma_W(\tilde{r} + \tilde{\alpha}(-r_0))$  and we can write

$$\begin{aligned} c_0 &= \sup_{r \in \text{cone } R} \sup_{\alpha > 0} \frac{\sigma_W(r) - \sigma_W(r + \alpha(-r_0))}{\alpha} \\ &\leq \sup_{r \in \text{cone } R} \sup_{\alpha > 0} \frac{\sigma_W(\alpha r_0)}{\alpha} \\ &= \sigma_W(r_0) \end{aligned}$$

where we have used the sublinearity of  $\sigma_W$  in the inequality and the second equality. The conclusion follows now from the fact that  $W$  is a finite set. ■

We define a sublinear function  $h'$  over  $\text{span } R$ :

$$\begin{aligned} h'(r) &:= \min_{\substack{c_0 x_0 + c^\top x \\ r_0 x_0 + R x = r, \\ x_0 \geq 0, x \geq 0.}} \end{aligned} \quad (6)$$

**Lemma 2.6.** The function  $h'$  coincides with  $h$  on  $\text{cone } R$ . Furthermore, for any  $r \in \text{cone } R$ , (6) admits an optimal solution of the form  $(0, x) \in \mathbb{R} \times \mathbb{R}^n$ .

*Proof.* It is clear that  $h' \leq h$ . Let  $\bar{r} \in \text{cone } R$  and suppose  $h'(\bar{r}) < h(\bar{r})$ . Then there exists  $(\bar{x}_0, \bar{x})$  satisfying  $r_0 \bar{x}_0 + R \bar{x} = \bar{r}$ ,  $\bar{x} \geq 0$ ,  $\bar{x}_0 > 0$  and  $c_0 \bar{x}_0 + c^\top \bar{x} < h(\bar{r})$ . Rearranging the terms and using Remark 2.2, we obtain

$$c_0 < \frac{h(\bar{r}) - c^\top \bar{x}}{\bar{x}_0} \leq \frac{h(\bar{r}) - \sum_{j=1}^n h(r_j) \bar{x}_j}{\bar{x}_0}.$$

Finally, the sublinearity of  $h$  and the observation that  $R \bar{x} = \bar{r} - r_0 \bar{x}_0$  give

$$c_0 < \frac{h(\bar{r}) - \sum_{j=1}^n h(\bar{x}_j r_j)}{\bar{x}_0} \leq \frac{h(\bar{r}) - h(R \bar{x})}{\bar{x}_0} = \frac{h(\bar{r}) - h(\bar{r} - r_0 \bar{x}_0)}{\bar{x}_0}.$$

This contradicts the definition of  $c_0$  and proves the first claim. Now let  $\tilde{x}$  be an optimal solution to (3) for  $r = \bar{r}$ . We have  $c^\top \tilde{x} = h(\bar{r}) = h'(\bar{r})$  and  $(0, \tilde{x})$  is feasible to (6). This shows that  $(0, \tilde{x})$  is an optimal solution to (6). ■

If  $\dim(R) < q$ , we extend the function  $h'$  defined in (6) to the whole of  $\mathbb{R}^q$  by letting

$$h'(r) = h'(r') \text{ for all } r \in \mathbb{R}^q, r' \in \text{span } R, r'' \in (\text{span } R)^\perp \text{ such that } r = r' + r''. \quad (7)$$

*Proof of Theorem 1.1.* Let  $h'$  be defined as in (6) and (7) and let  $V' := \{r \in \mathbb{R}^q : h'(r) \leq 1\}$ . Observe that  $V'$  is a closed convex neighborhood of the origin because  $h'$  is sublinear and finite everywhere. Furthermore,  $\text{int}(V') = \{r \in \mathbb{R}^q : h'(r) < 1\}$  by the Slater property.

*Claim 2.2:*  $V'$  is  $S$ -free. Suppose this is not the case. Let  $\bar{r} \in S$  be a point in  $\text{int}(V')$ . By Lemma 2.6, there exists  $\bar{x} \geq 0$  such that  $R\bar{x} = \bar{r} \in S$  and  $c^\top \bar{x} = h'(\bar{r}) < 1$ . Because  $\bar{x} \in X$ , this contradicts the validity of  $c^\top x \geq 1$ . □

Now, by Remark 2.2 and Lemma 2.6,  $h'(r_j) = h(r_j) \leq c_j$  for all  $j \in [n]$ . This shows that the  $S$ -intersection cut  $\sum_{j=1}^n h'(r_j)x_j \geq 1$  dominates  $c^\top x \geq 1$ . ■

### 3 Constructing the $S$ -Free Convex Neighborhood of the Origin

We now give a geometric interpretation for the proof of Theorem 1.1. Again let  $c^\top x \geq 1$  be a valid inequality separating the origin from  $X$ . Assume without any loss of generality that the vectors  $r_1, \dots, r_n$  have been normalized so that  $c_j \in \{0, \pm 1\}$  for all  $j \in [n]$ . Define the sets  $J_+ := \{j \in [n] : c_j = +1\}$ ,  $J_- := \{j \in [n] : c_j = -1\}$  and  $J_0 := \{j \in [n] : c_j = 0\}$ . Let  $C := \text{conv}(\{0\} \cup \{r_j : j \in J_+\})$  and  $K := \text{cone}(\{r_j : j \in J_0 \cup J_-\} \cup \{r_j + r_i : j \in J_+, i \in J_-\})$ . Let  $Q := C + K$  and  $h$  be defined as in (3). One can show  $Q = \{r \in \mathbb{R}^q : h(r) \leq 1\}$ . However, when  $\text{cone } R \neq \mathbb{R}^q$ , the origin lies on the boundary of  $Q$ . In the proof of Theorem 1.1, we overcame this difficulty by extending  $h$  into a function  $h'$  which is defined on the whole of  $\mathbb{R}^q$  and coincides with  $h$  on  $\text{cone } R$ . We can also follow a similar approach here. Let  $r_0 \in -\text{ri}(\text{cone } R)$  and let  $c_0$  be as defined in (5). When  $c_0 \neq 0$ , scale  $r_0$  so that  $c_0 \in \{\pm 1\}$ . Introduce  $r_0$  into the relevant subset of  $[n]$  according to the sign of  $c_0$ : If  $c_0 = +1$ , let  $J'_+ := J_+ \cup \{0\}$ ,  $J'_0 := J_0$  and  $J'_- := J_-$ ; otherwise, if  $c_0 = 0$ , let  $J'_+ := J_+$ ,  $J'_0 := J_0 \cup \{0\}$  and  $J'_- := J_-$ ; otherwise ( $c_0 = -1$ ), let  $J'_+ := J_+$ ,  $J'_0 := J_0$  and  $J'_- := J_- \cup \{0\}$ . Finally, let

$C' := \text{conv}(\{0\} \cup \{r_j : j \in J'_+\})$ ,  $K' := \text{cone}(\{r_j : j \in J'_0 \cup J'_-\} \cup \{r_j + r_i : j \in J'_+, i \in J'_-\})$  and  $Q' := C' + K' + (\text{span } R)^\perp$ . The following proposition shows that  $h'$  represents  $Q'$  and  $Q'$  can be used to generate an  $S$ -intersection cut that dominates  $c^\top x \geq 1$ .

**Proposition 3.1.**  $Q' = \{r \in \mathbb{R}^q : h'(r) \leq 1\}$  where  $h'$  is defined as in (6) and (7).

*Proof.* Let  $V' := \{r \in \mathbb{R}^q : h'(r) \leq 1\}$ . Note that  $V'$  is convex by the sublinearity of  $h'$ . We have  $h'(r_j) \leq c_j = 1$  for all  $j \in J'_+$ ,  $h'(r_j) \leq c_j \leq 0$  for all  $j \in J'_0 \cup J'_-$  and  $h'(r_j + r_i) \leq h'(r_j) + h'(r_i) \leq c_j + c_i = 0$  for all  $j \in J'_+$  and  $i \in J'_-$ . Moreover,  $h'(r) = h'(r + r')$  for all  $r \in \mathbb{R}^q$  and  $r' \in (\text{span } R)^\perp$  by the definition of  $h'$ . Hence  $C' \subseteq V'$ ,  $K' \subseteq \text{rec}(V')$  and  $(\text{span } R)^\perp \subseteq \text{lin}(V')$  which together give us  $Q' = C' + K' + (\text{span } R)^\perp \subseteq V'$ .

To prove the converse, let  $\bar{r} \in \mathbb{R}^q$  be such that  $h'(\bar{r}) \leq 1$ . We consider two distinct cases:  $h'(\bar{r}) \leq 0$  and  $0 < h'(\bar{r}) \leq 1$ . First, let us suppose  $h'(\bar{r}) \leq 0$ . Then the definition of  $h'$  implies that there exist  $(\bar{x}_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n$  and  $\bar{r}' \in (\text{span } R)^\perp$  such that  $(\bar{x}_0, \bar{x}) \geq 0$ ,  $\sum_{j \in J'_+} \bar{x}_j - \sum_{i \in J'_-} \bar{x}_i \leq 0$  and  $r_0 \bar{x}_0 + R\bar{x} = \bar{r} - \bar{r}'$ . It can be verified by inspection that the first two sets of inequalities define a cone generated by the rays  $\{e_j : j \in J'_0 \cup J'_-\} \cup \{e_j + e_i : j \in J'_+, i \in J'_-\}$ . This shows  $\bar{r} \in K' + (\text{span } R)^\perp \subseteq Q'$ . Now suppose  $0 < h'(\bar{r}) \leq 1$ . Then there exist  $(\bar{x}_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^n$  and  $\bar{r}' \in (\text{span } R)^\perp$  such that  $(\bar{x}_0, \bar{x}) \geq 0$ ,  $0 < \sum_{j \in J'_+} \bar{x}_j - \sum_{i \in J'_-} \bar{x}_i \leq 1$  and  $r_0 \bar{x}_0 + R\bar{x} = \bar{r} - \bar{r}'$ . Define  $\bar{x}_i^j := \bar{x}_i \frac{\bar{x}_j}{\sum_{j \in J'_+} \bar{x}_j}$  for all  $i \in J'_-$  and  $j \in J'_+$ . These values are well-defined since  $0 \leq \sum_{i \in J'_-} \bar{x}_i < \sum_{j \in J'_+} \bar{x}_j$ . Observe that  $\sum_{j \in J'_+} \bar{x}_i^j = \bar{x}_i$  and  $r_0 \bar{x}_0 + R\bar{x} = \sum_{j \in J'_+} (\bar{x}_j - \sum_{i \in J'_-} \bar{x}_i^j) r_j + \sum_{i \in J'_-} \sum_{j \in J'_+} \bar{x}_i^j (r_i + r_j) + \sum_{j \in J'_0} \bar{x}_j r_j$ . We have  $\sum_{j \in J'_+} (\bar{x}_j - \sum_{i \in J'_-} \bar{x}_i^j) = \sum_{j \in J'_+} \bar{x}_j - \sum_{i \in J'_-} \bar{x}_i \leq 1$  together with  $\bar{x}_j - \sum_{i \in J'_-} \bar{x}_i^j > 0$  which is true for all  $j \in J'_+$  because  $\sum_{i \in J'_-} \bar{x}_i^j = \bar{x}_j \frac{\sum_{i \in J'_-} \bar{x}_i}{\sum_{j \in J'_+} \bar{x}_j} < \bar{x}_j$ . Hence  $\sum_{j \in J'_+} (\bar{x}_j - \sum_{i \in J'_-} \bar{x}_i^j) r_j \in C'$ . Moreover,  $\sum_{i \in J'_-} \sum_{j \in J'_+} \bar{x}_i^j (r_i + r_j) + \sum_{j \in J'_0} \bar{x}_j r_j \in K'$ . These yield  $\bar{r} \in C' + K' + (\text{span } R)^\perp = Q'$ . ■

The proof of Theorem 1.1 shows that  $V' := \{r \in \mathbb{R}^q : h'(r) \leq 1\}$  is a closed convex  $S$ -free neighborhood of the origin. Proposition 3.1 shows that  $Q' = V'$ . Therefore,  $\sum_{j=1}^n h'(r_j) x_j \geq 1$  is an  $S$ -intersection cut obtained from  $Q'$ .



## References

- [ALWW07] K. Andersen, Q. Louveaux, R. Weismantel, and L.A. Wolsey. Cutting planes from two rows of a simplex tableau. In *Proceedings of IPCO XII*, volume 4513 of *Lecture Notes in Computer Science*, pages 1–15, Ithaca, New York, June 2007.
- [CCD<sup>+</sup>13] M. Conforti, G. Cornuéjols, A. Daniilidis, C. Lemaréchal, and J. Malick. Cut-generating functions and  $S$ -free sets. February 2013. Working Paper.
- [CCZ10] M. Conforti, G. Cornuéjols, and G. Zambelli. Equivalence between intersection cuts and the corner polyhedron. *Operations Research Letters*, 38:153–155, 2010.
- [GJ72] R.E. Gomory and E.L. Johnson. Some continuous functions related to corner polyhedra. *Mathematical Programming*, 3:23–85, 1972.
- [Gom69] R.G. Gomory. Some polyhedra related to combinatorial problems. *Linear Algebra and Applications*, 2:451–558, 1969.
- [HUL04] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Grundlehren Text Editions. Springer, Berlin, 2004.
- [Joh81] E.L. Johnson. Characterization of facets for multiple right-hand side choice linear programs. *Mathematical Programming Study*, 14:112–142, 1981.
- [JSRF06] J.J. Júdice, H. Sherali, I.M. Ribeiro, and A.M. Faustino. A complementarity-based partitioning and disjunctive cut algorithm for mathematical programming problems with equilibrium constraints. *Journal of Global Optimization*, 136:89–114, 2006.
- [Roc70] R.T. Rockafellar. *Convex Analysis*. Princeton Landmarks in Mathematics. Princeton University Press, New Jersey, 1970.

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